Unit V:

Graphs: Graphs – Representation of graph – Traversal in Graph – Spanning Trees - Prim's and Kruskal's algorithm – Dijkstra's algorithm for shortest path problem.

Introduction to Graphs

- Graph is a non-linear data structure.
- It contains a set of points known as nodes (or vertices) and a set of links known as edges (or Arcs). Here edges are used to connect the vertices.

A graph is defined as follows...

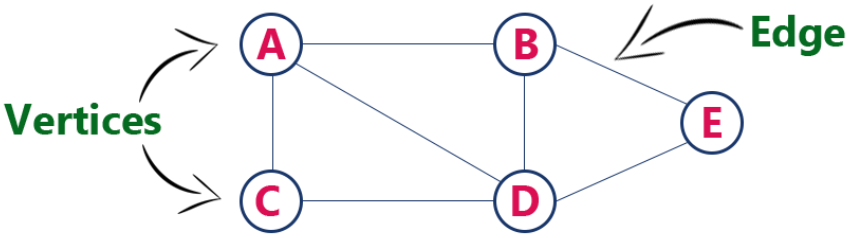
Graph is a collection of vertices and arcs in which vertices are connected with arcs

Graph is a collection of nodes and edges in which nodes are connected with edges

Generally, a graph **G** is represented as **G = (V , E)**, where **V is set of vertices** and **E is set of edges**.

Example

The following is a graph with 5 vertices and 6 edges. This graph G can be defined as G = (V , E) Where V = {A,B,C,D,E} and E = {(A,B),(A,C)(A,D),(B,D),(C,D),(B,E),(E,D)}.



Graph Representations

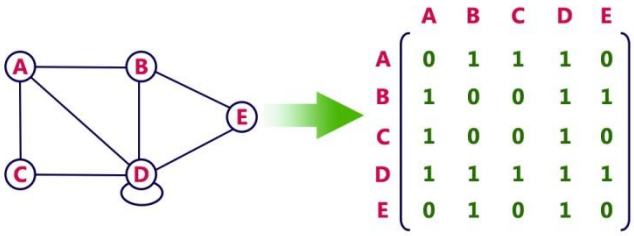
Graph data structure is represented using following representations...

1. **Adjacency Matrix**
2. **Incidence Matrix**
3. **Adjacency List**

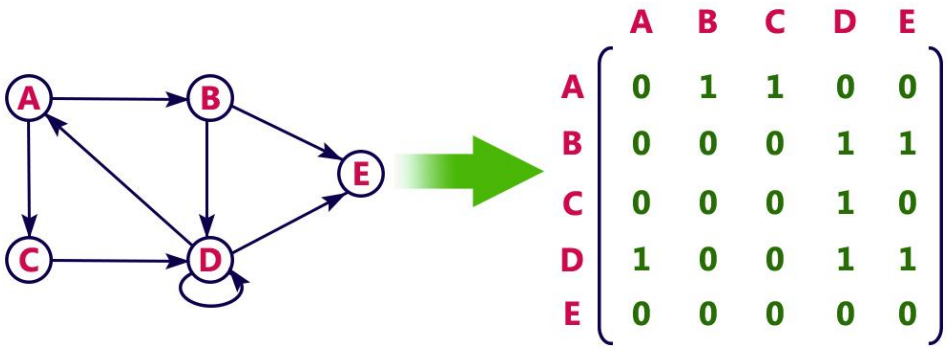
Adjacency Matrix

- In this representation, the graph is represented using a matrix of size total number of vertices by a total number of vertices.
- That means a graph with 4 vertices is represented using a matrix of size 4X4. In this matrix, both rows and columns represent vertices.
- This matrix is filled with either 1 or 0.
- Here, 1 represents that there is an edge from row vertex to column vertex and 0 represents that there is no edge from row vertex to column vertex.

For example, consider the following undirected graph representation...



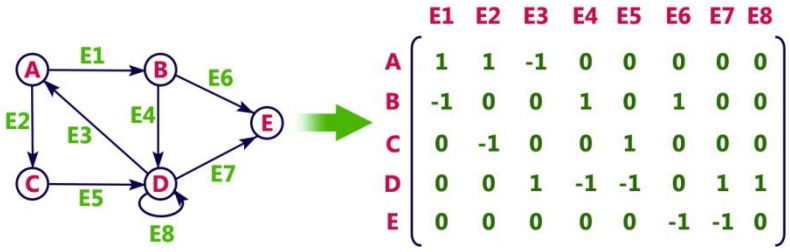
Directed graph representation...



Incidence Matrix

- In this representation, the graph is represented using a matrix of size total number of vertices by a total number of edges.
- That means graph with 4 vertices and 6 edges is represented using a matrix of size 4X6. In this matrix, rows represent vertices and columns represents edges.
- This matrix is filled with 0 or 1 or -1.
- Here, 0 represents that the row edge is not connected to column vertex, 1 represents that the row edge is connected as the outgoing edge to column vertex and -1 represents that the row edge is connected as the incoming edge to column vertex.

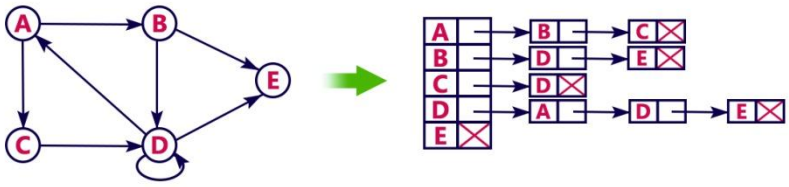
  For example, consider the following directed graph representation...



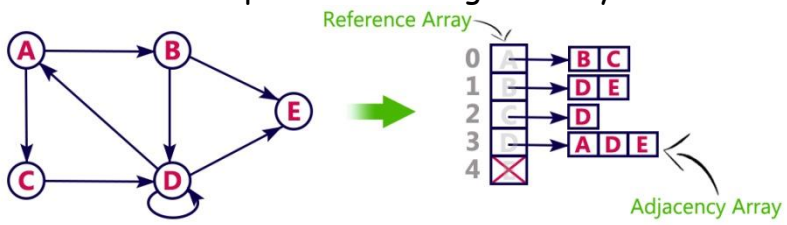
Adjacency List

In this representation, every vertex of a graph contains list of its adjacent vertices.

For example, consider the following directed graph representation implemented using linked list...



This representation can also be implemented using an array as follows..



Graph Traversal - DFS

- Graph traversal is a technique used for a searching vertex in a graph. The graph traversal is also used to decide the order of vertices is visited in the search process.
- A graph traversal finds the edges to be used in the search process without creating loops.
- That means using graph traversal we visit all the vertices of the graph without getting into looping path.

There are two graph traversal techniques and they are as follows...

1. **DFS (Depth First Search)**
2. **BFS (Breadth First Search)**

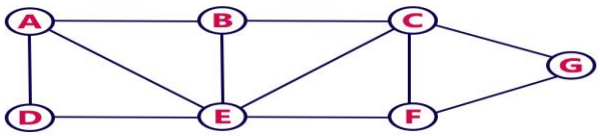
DFS (Depth First Search)

DFS traversal of a graph produces a **spanning tree** as final result. **Spanning Tree** is a graph without loops. We use **Stack data structure** with maximum size of total number of vertices in the graph to implement DFS traversal.

We use the following steps to implement DFS traversal...

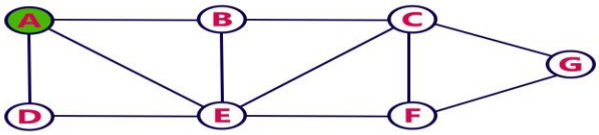
- **Step 1 -** Define a Stack of size total number of vertices in the graph.
- **Step 2 -** Select any vertex as **starting point** for traversal. Visit that vertex and push it on to the Stack.
- **Step 3 -** Visit any one of the non-visited **adjacent** vertices of a vertex which is at the top of stack and push it on to the stack.
- **Step 4 -** Repeat step 3 until there is no new vertex to be visited from the vertex which is at the top of the stack.
- **Step 5 -** When there is no new vertex to visit then use **back tracking** and pop one vertex from the stack.
- **Step 6 -** Repeat steps 3, 4 and 5 until stack becomes Empty.
- **Step 7 -** When stack becomes Empty, then produce final spanning tree by removing unused edges from the graph

Consider the following example graph to perform DFS traversal



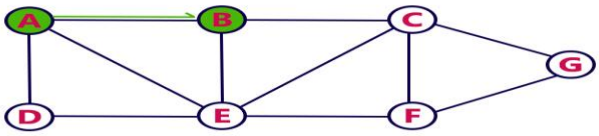
Step 1:

- Select the vertex **A** as starting point (visit **A**).
- Push **A** on to the Stack.



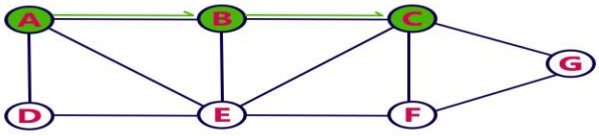
Step 2:

- Visit any adjacent vertex of **A** which is not visited (**B**).
- Push newly visited vertex B on to the Stack.



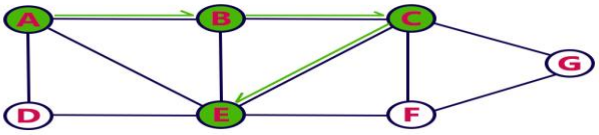
Step 3:

- Visit any adjacent vertext of **B** which is not visited (**C**).
- Push C on to the Stack.



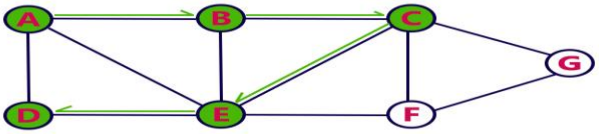
Step 4:

- Visit any adjacent vertext of **C** which is not visited (**E**).
- Push E on to the Stack



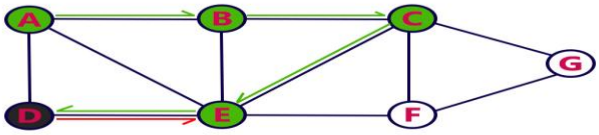
Step 5:

- Visit any adjacent vertext of **E** which is not visited (**D**).
- Push D on to the Stack



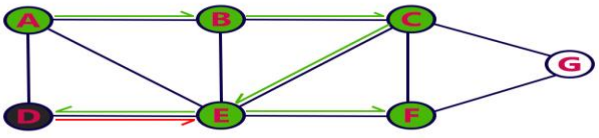
Step 6:

- There is no new vertiex to be visited from D. So use back track.
- Pop D from the Stack.



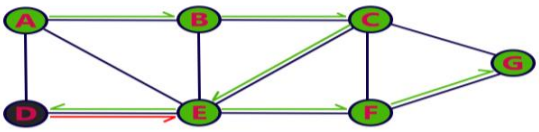
Step 7:

- Visit any adjacent vertex of **E** which is not visited (**F**).
- Push **F** on to the Stack.



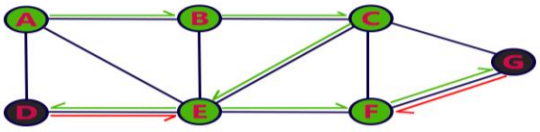
Step 8:

- Visit any adjacent vertex of **F** which is not visited (**G**).
- Push **G** on to the Stack.



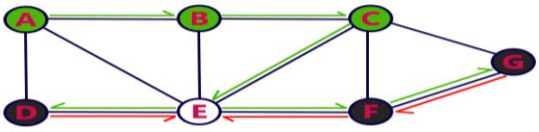
Step 9:

- There is no new vertiex to be visited from G. So use back track.
- Pop G from the Stack.



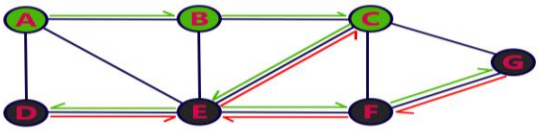
Step 10:

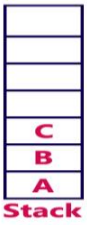
- There is no new vertiex to be visited from F. So use back track.
- Pop F from the Stack.



Step 11:

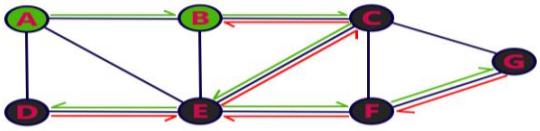
- There is no new vertiex to be visited from E. So use back track.
- Pop E from the Stack.





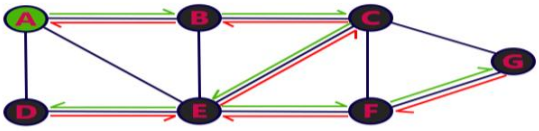
Step 12:

- There is no new vertiex to be visited from C. So use back track.
- Pop C from the Stack.



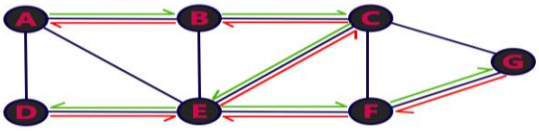
Step 13:

- There is no new vertiex to be visited from B. So use back track.
- Pop B from the Stack.

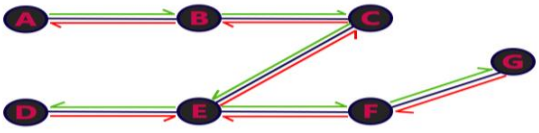


Step 14:

- There is no new vertiex to be visited from A. So use back track.
- Pop A from the Stack.



- Stack became Empty. So stop DFS Treversal.
- Final result of DFS traversal is following spanning tree.



BFS (Breadth First Search)

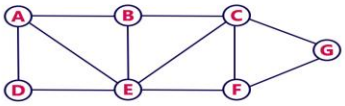
BFS traversal of a graph produces a **spanning tree** as final result. **Spanning Tree** is a graph without loops. We use **Queue data structure** with maximum size of total number of vertices in the graph to implement BFS traversal.

We use the following steps to implement BFS traversal...

- **Step 1 -** Define a Queue of size total number of vertices in the graph.
- **Step 2 -** Select any vertex as **starting point** for traversal. Visit that vertex and insert it into the Queue.
- **Step 3 -** Visit all the non-visited **adjacent** vertices of the vertex which is at front of the Queue and insert them into the Queue.
- **Step 4 -** When there is no new vertex to be visited from the vertex which is at front of the Queue then delete that vertex.
- **Step 5 -** Repeat steps 3 and 4 until queue becomes empty.
- **Step 6 -** When queue becomes empty, then produce final spanning tree by removing unused edges from the graph

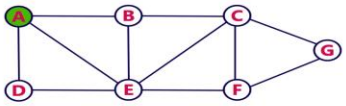
Example

Consider the following example graph to perform BFS traversal



Step 1:

- Select the vertex **A** as starting point (visit **A**).
- Insert **A** into the Queue.

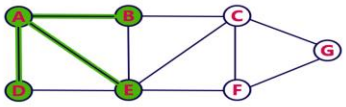


Queue

A						

Step 2:

- Visit all adjacent vertices of **A** which are not visited (**D**, **E**, **B**).
- Insert newly visited vertices into the Queue and delete A from the Queue..

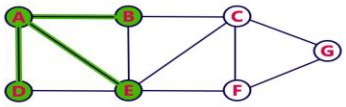


Queue

	D	E	B			

Step 3:

- Visit all adjacent vertices of **D** which are not visited (there is no vertex).
- Delete D from the Queue.

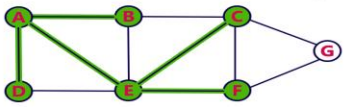


Queue

		E	B			

Step 4:

- Visit all adjacent vertices of **E** which are not visited (**C**, **F**).
- Insert newly visited vertices into the Queue and delete E from the Queue.

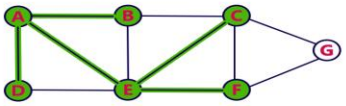


Queue

			B	C	F	

Step 5:

- Visit all adjacent vertices of **B** which are not visited (**there is no vertex**).
- Delete **B** from the Queue.

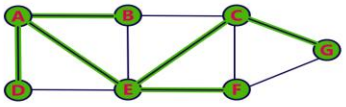


Queue

				C	F	

Step 6:

- Visit all adjacent vertices of **C** which are not visited (**G**).
- Insert newly visited vertex into the Queue and delete **C** from the Queue.

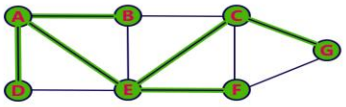


Queue

					F	G

Step 7:

- Visit all adjacent vertices of **F** which are not visited (**there is no vertex**).
- Delete **F** from the Queue.

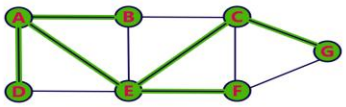


Queue

						G

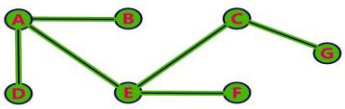
Step 8:

- Visit all adjacent vertices of **G** which are not visited (**there is no vertex**).
- Delete **G** from the Queue.



Queue

- Queue became Empty. So, stop the BFS process.
- Final result of BFS is a Spanning Tree as shown below...



What is a spanning tree?

A spanning tree can be defined as the subgraph of an undirected connected graph. It includes all the vertices along with the least possible number of edges. If any vertex is missed, it is not a spanning tree. A spanning tree is a subset of the graph that does not have cycles, and it also cannot be disconnected.

A spanning tree consists of (n-1) edges, where 'n' is the number of vertices (or nodes). Edges of the spanning tree may or may not have weights assigned to them.
All the possible spanning trees created from the given graph G would have the same number of vertices, but the number of edges in the spanning tree would be equal to the number of vertices in the given graph minus 1.

A complete undirected graph can have $\mathbf{n^{n-2}}$ number of spanning trees where **n** is the number of vertices in the graph. Suppose, if $\mathbf{n = 5}$, the number of maximum possible spanning trees would be $\mathbf{5^{5-2} = 125}$.

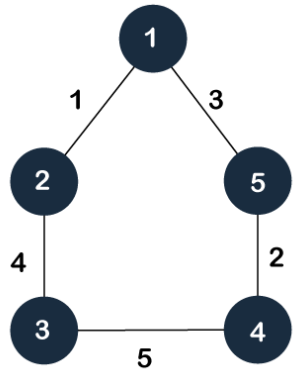
Applications of the spanning tree

Basically, a spanning tree is used to find a minimum path to connect all nodes of the graph. Some of the common applications of the spanning tree are listed as follows -

- Cluster Analysis
- Civil network planning
- Computer network routing protocol

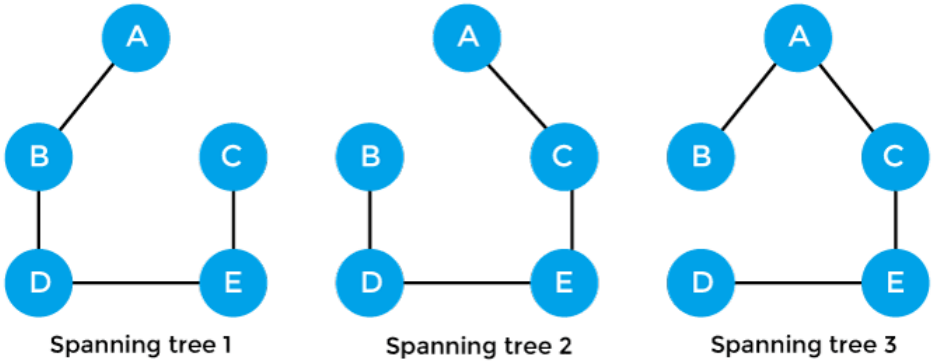
Example of Spanning tree

Suppose the graph be -



As discussed above, a spanning tree contains the same number of vertices as the graph, the number of vertices in the above graph is 5; therefore, the spanning tree will contain 5 vertices. The edges in the spanning tree will be equal to the number of vertices in the graph minus 1. So, there will be 4 edges in the spanning tree.

Some of the possible spanning trees that will be created from the above graph are given as follows -



Properties of spanning-tree

Some of the properties of the spanning tree are given as follows -

- There can be more than one spanning tree of a connected graph G.
- A spanning tree does not have any cycles or loop.
- A spanning tree is **minimally connected**, so removing one edge from the tree will make the graph disconnected.
- A spanning tree is **maximally acyclic**, so adding one edge to the tree will create a loop.
- There can be a maximum $\mathbf{n^{n-2}}$ number of spanning trees that can be created from a complete graph.
- A spanning tree has **n-1** edges, where 'n' is the number of nodes.
- If the graph is a complete graph, then the spanning tree can be constructed by removing maximum (e-n+1) edges, where 'e' is the number of edges and 'n' is the number of vertices.

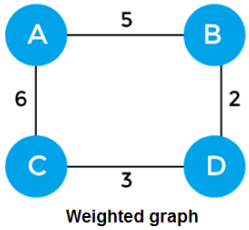
So, a spanning tree is a subset of connected graph G, and there is no spanning tree of a disconnected graph.

Minimum Spanning tree

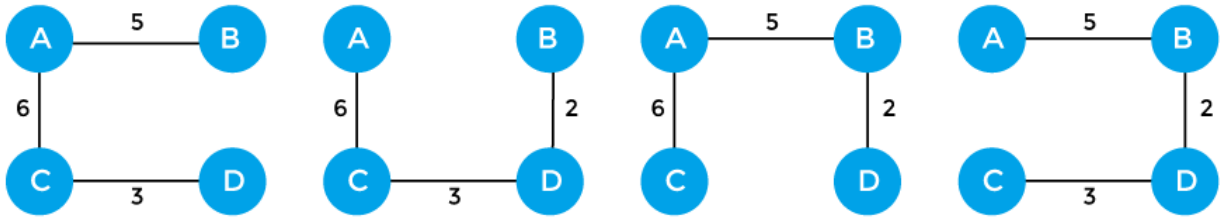
A minimum spanning tree can be defined as the spanning tree in which the sum of the weights of the edge is minimum. The weight of the spanning tree is the sum of the weights given to the edges of the spanning tree. In the real world, this weight can be considered as the distance, traffic load, congestion, or any random value.

Example of minimum spanning tree

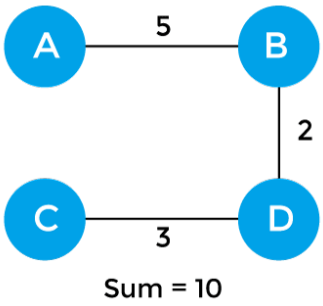
Let's understand the minimum spanning tree with the help of an example.



The sum of the edges of the above graph is 16. Now, some of the possible spanning trees created from the above graph are -



So, the minimum spanning tree that is selected from the above spanning trees for the given weighted graph is -



Applications of minimum spanning tree

The applications of the minimum spanning tree are given as follows -

- Minimum spanning tree can be used to design water-supply networks, telecommunication networks, and electrical grids.
- It can be used to find paths in the map.

Algorithms for Minimum spanning tree

A minimum spanning tree can be found from a weighted graph by using the algorithms given below -

- Prim's Algorithm
- Kruskal's Algorithm

Prim's Algorithm

Prim's Algorithm is a greedy algorithm that is used to find the minimum spanning tree from a graph. Prim's algorithm finds the subset of edges that includes every vertex of the graph such that the sum of the weights of the edges can be minimized.

Prim's algorithm starts with the single node and explores all the adjacent nodes with all the connecting edges at every step. The edges with the minimal weights causing no cycles in the graph got selected.

How does the prim's algorithm work?

Prim's algorithm is a greedy algorithm that starts from one vertex and continue to add the edges with the smallest weight until the goal is reached. The steps to implement the prim's algorithm are given as follows -

- First, we have to initialize an MST with the randomly chosen vertex.

- Now, we have to find all the edges that connect the tree in the above step with the new vertices. From the edges found, select the minimum edge and add it to the tree.
- Repeat step 2 until the minimum spanning tree is formed.

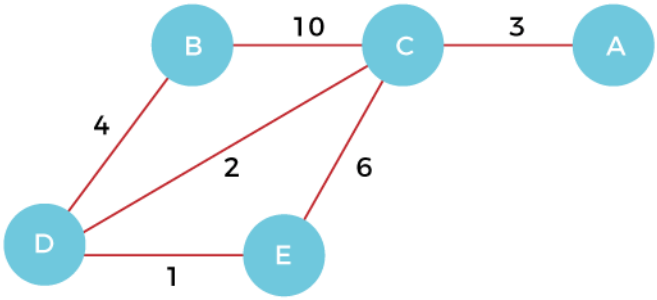
The applications of prim's algorithm are -

- Prim's algorithm can be used in network designing.
- It can be used to make network cycles.
- It can also be used to lay down electrical wiring cables.

Example of prim's algorithm

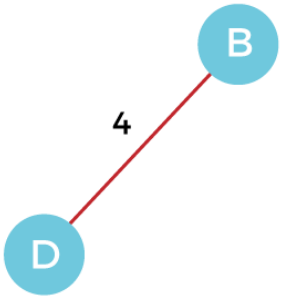
Now, let's see the working of prim's algorithm using an example. It will be easier to understand the prim's algorithm using an example.

Suppose, a weighted graph is -

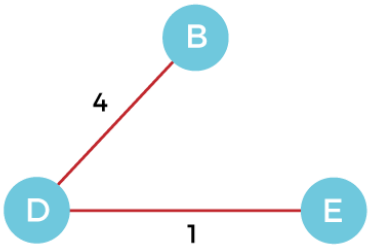


Step 1 - First, we have to choose a vertex from the above graph. Let's choose B.

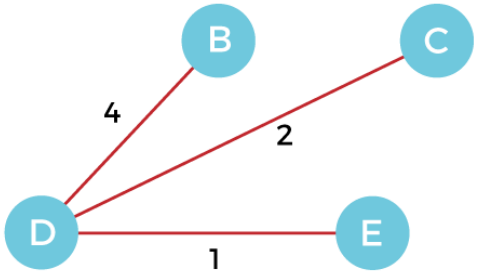
Step 2 - Now, we have to choose and add the shortest edge from vertex B. There are two edges from vertex B that are B to C with weight 10 and edge B to D with weight 4. Among the edges, the edge BD has the minimum weight. So, add it to the MST.



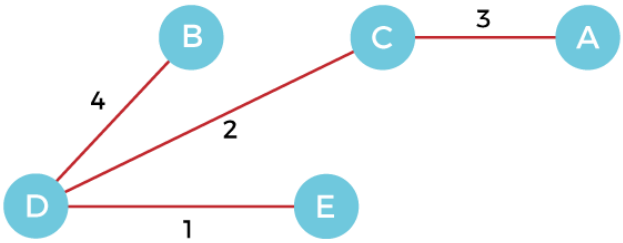
Step 3 - Now, again, choose the edge with the minimum weight among all the other edges. In this case, the edges DE and CD are such edges. Add them to MST and explore the adjacent of C, i.e., E and A. So, select the edge DE and add it to the MST.



Step 4 - Now, select the edge CD, and add it to the MST.



Step 5 - Now, choose the edge CA. Here, we cannot select the edge CE as it would create a cycle to the graph. So, choose the edge CA and add it to the MST.



So, the graph produced in step 5 is the minimum spanning tree of the given graph. The cost of the MST is given below -

Cost of MST = 4 + 2 + 1 + 3 = 10 units.

Algorithm

1. Step 1: Select a starting vertex
2. Step 2: Repeat Steps 3 and 4 until there are fringe vertices
3. Step 3: Select an edge 'e' connecting the tree vertex and fringe vertex that ha s minimum weight
4. Step 4: Add the selected edge and the vertex to the minimum spanning tree T
5. [END OF LOOP]
6. Step 5: EXIT

Kruskal's Algorithm

Kruskal's Algorithm is used to find the minimum spanning tree for a connected weighted graph. The main target of the algorithm is to find the subset of edges by using which we can traverse every vertex of the graph. It follows the greedy approach that finds an optimum solution at every stage instead of focusing on a global optimum.

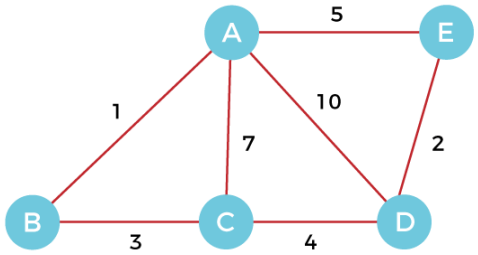
How does Kruskal's algorithm work?

In Kruskal's algorithm, we start from edges with the lowest weight and keep adding the edges until the goal is reached. The steps to implement Kruskal's algorithm are listed as follows -

- First, sort all the edges from low weight to high.
- Now, take the edge with the lowest weight and add it to the spanning tree. If the edge to be added creates a cycle, then reject the edge.
- Continue to add the edges until we reach all vertices, and a minimum spanning tree is created.

The applications of Kruskal's algorithm are -

- Kruskal's algorithm can be used to layout electrical wiring among cities.
- It can be used to lay down LAN connections.
- Example of Kruskal's algorithm
- Now, let's see the working of Kruskal's algorithm using an example. It will be easier to understand Kruskal's algorithm using an example.
- Suppose a weighted graph is -



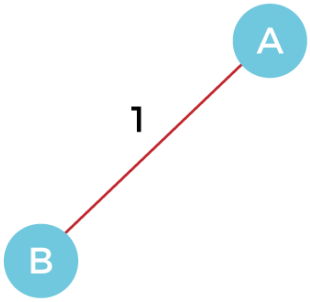
- The weight of the edges of the above graph is given in the below table -

Edge	AB	AC	AD	AE	BC	CD	DE
Weight	1	7	10	5	3	4	2

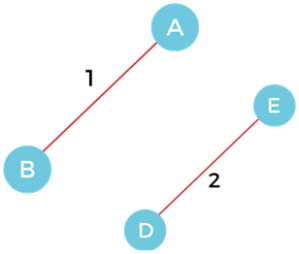
- Now, sort the edges given above in the ascending order of their weights.

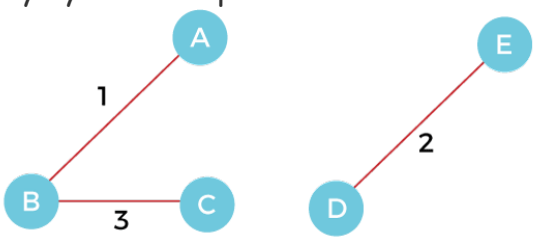
Edge	AB	DE	BC	CD	AE	AC	AD
Weight	1	2	3	4	5	7	10

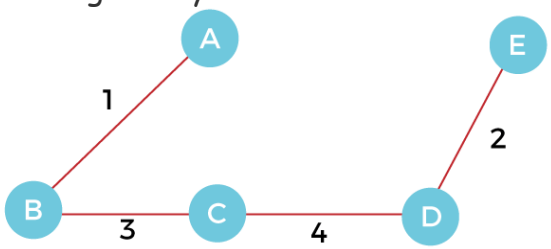
- Now, let's start constructing the minimum spanning tree.
- **Step 1** - First, add the edge **AB** with weight **1** to the MST.

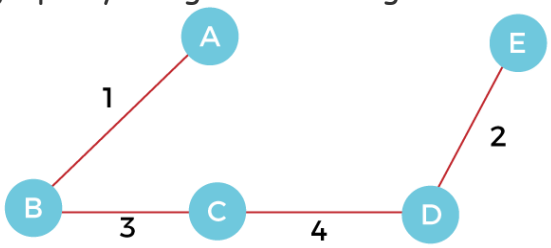


- **Step 2** - Add the edge **DE** with weight **2** to the MST as it is not creating the cycle.



- **Step 3** - Add the edge **BC** with weight **3** to the MST, as it is not creating any cycle or loop.
- 

- **Step 4** - Now, pick the edge **CD** with weight **4** to the MST, as it is not forming the cycle.
- 

- **Step 5** - After that, pick the edge **AE** with weight **5**. Including this edge will create the cycle, so discard it.
- **Step 6** - Pick the edge **AC** with weight **7**. Including this edge will create the cycle, so discard it.
- **Step 7** - Pick the edge **AD** with weight **10**. Including this edge will also create the cycle, so discard it.
- So, the final minimum spanning tree obtained from the given weighted graph by using Kruskal's algorithm is -
- 

Now, the number of edges in the above tree equals the number of vertices minus 1. So, the algorithm stops here.

Algorithm

- Step 1: Create a forest F in such a way that every vertex of the graph is a separate tree.
- Step 2: Create a set E that contains all the edges of the graph.
- Step 3: Repeat Steps 4 and 5 **while** E is NOT EMPTY and F is not spanning
- Step 4: Remove an edge from E with minimum weight
- Step 5: IF the edge obtained in Step 4 connects two different trees, the n add it to the forest F
- (**for** combining two trees into one tree).
- ELSE
- Discard the edge
- Step 6: END

Complexity of Kruskal's algorithm

Now, let's see the time complexity of Kruskal's algorithm.

- **Time Complexity**
- The time complexity of Kruskal's algorithm is O(E logE) or O(V logV), where E is the no. of edges, and V is the no. of vertices.

Implementation of Kruskal's algorithm

Now, let's see the implementation of kruskal's algorithm.

Program: Write a program to implement kruskal's algorithm in C++.

```
#include <iostream>
#include <algorithm>
using namespace std;
const int MAX = 1e4 + 5;
int id[MAX], nodes, edges;
pair <long long, pair<int, int> > p[MAX];
void init()
{
    for(int i = 0;i < MAX;++i)
        id[i] = i;
}
int root(int x)
{
    while(id[x] != x)
    {
        id[x] = id[id[x]];
        x = id[x];
    }
    return x;
}
void union1(int x, int y)
{
    int p = root(x);
    int q = root(y);
    id[p] = id[q];
}
long long kruskal(pair<long long, pair<int, int> > p[])
{
    int x, y;
    long long cost, minimumCost = 0;
    for(int i = 0;i < edges;++i)
    {
        x = p[i].second.first;
        y = p[i].second.second;
        cost = p[i].first;
        if(root(x) != root(y))
```

```
37.      {
38.          minimumCost += cost;
39.          union1(x, y);
40.      }
41.   }
42.   return minimumCost;
43.}
44.int main()
45.{
46.   int x, y;
47.   long long weight, cost, minimumCost;
48.   init();
49.   cout <<"Enter Nodes and edges";
50.   cin >> nodes >> edges;
51.   for(int i = 0;i < edges;++i)
52.   {
53.      cout<<"Enter the value of X, Y and edges";
54.   cin >> x >> y >> weight;
55.      p[i] = make_pair(weight, make_pair(x, y));
56.   }
57.   sort(p, p + edges);
58.   minimumCost = kruskal(p);
59.   cout <<"Minimum cost is "<< minimumCost << endl;
60.   return 0;
61.}
```

Output

```
Enter Nodes and edges 5 7
Enter the value of X, Y and edges 1 2 1
Enter the value of X, Y and edges 1 3 7
Enter the value of X, Y and edges 1 4 10
Enter the value of X, Y and edges 1 5 5
Enter the value of X, Y and edges 2 3 3
Enter the value of X, Y and edges 3 4 4
Enter the value of X, Y and edges 4 5 2
Minimum cost is 10
```

Dijkstra's Algorithm

Djikstra used this property in the opposite direction i.e we overestimate the distance of each vertex from the starting vertex. Then we visit each node and its neighbors to find the shortest subpath to those neighbors.
The algorithm uses a greedy approach in the sense that we find the next best solution hoping that the end result is the best solution for the whole problem.

Dijkstra's Algorithm Complexity

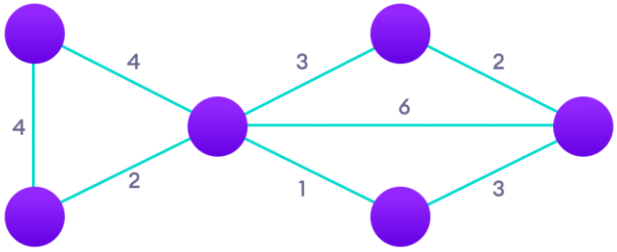
Time Complexity: O(E Log V)
where, E is the number of edges and V is the number of vertices.
Space Complexity: O(V)

Dijkstra's Algorithm Applications

- To find the shortest path
- In social networking applications
- In a telephone network
- To find the locations in the map

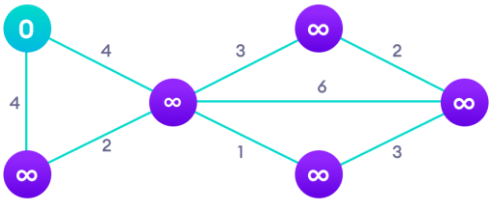
Example of Dijkstra's algorithm

It is easier to start with an example and then think about the algorithm.



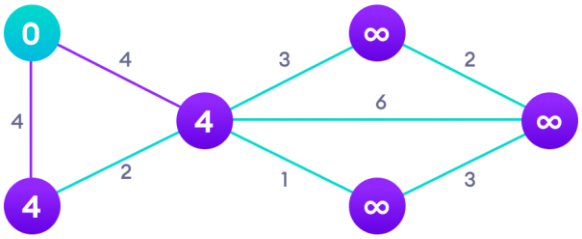
Step: 1

Start with a weighted graph



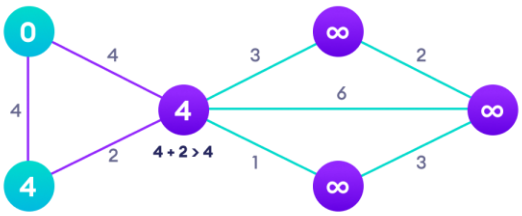
Step: 2

Choose a starting vertex and assign infinity path values to all other devices



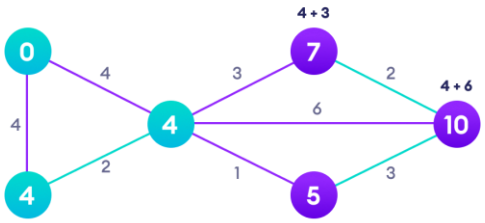
Step: 3

Go to each vertex and update its path length



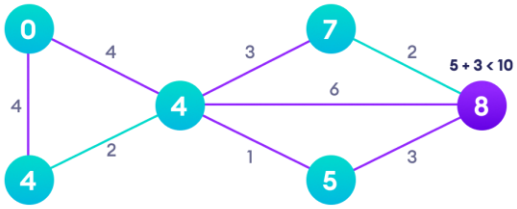
Step: 4

If the path length of the adjacent vertex is lesser than new path length, don't



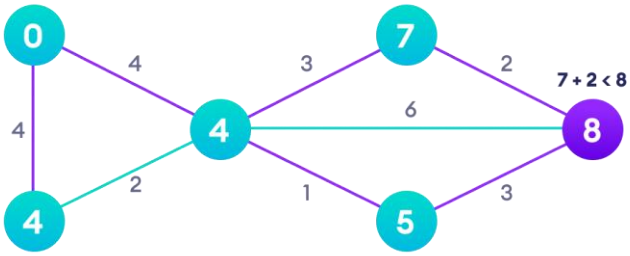
Step: 5

update it
Avoid updating path lengths of already visited vertices



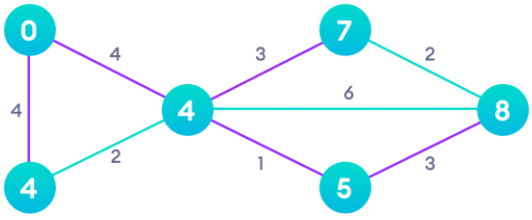
Step: 6

After each iteration, we pick the unvisited vertex with the least path length. So we choose 5 before 7



Step: 7

Notice how the rightmost vertex has its path length updated twice



Step: 8

Repeat until all the vertices have been visited